Non-Existence of Optimal Programs in Continuous Time

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Giorgio Fabbri[1], Silvia Faggian[2] and Giuseppe Freni[3]

We report an example of a two-dimensional undiscounted convex optimal growth model in continuous time in which, although there is a unique "golden rule", no overtaking optimal solutions exists in a full neighborhood of the steady state. The example proves, for optimal growth models, a conjecture advanced in 1976 by Brock and Haurie that the minimum dimension for non-existence of overtaking optimal programs in continuous time is 2.

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1. INTRODUCTION

For the class of undiscounted convex models of optimal growth, it has been known since Gale (1967) that existence of optimal (in the sense of overtaking) solutions cannot be proved in general if the "golden rule" capital stock is not unique. Soon, however, it turned out that uniqueness is not sufficient for the existence of an optimal solution. Brock (1970), indeed, proved existence under this condition, but used the weaker optimality criterion know as maximality (or weak overtaking optimality) and presented an example of a maximal steady state that is not optimal. Peleg (1973) then pointed out that the same example can be used to prove non-existence of optimal paths, implying that, without additional assumptions, it is not possible to strength Brock's existence theorem.

There are only few published examples of non-existence: the Brock-Peleg one, one reported in Khan & Piazza (2010), one contained in a paper by Leizarowitz (1985) and finally the one provided in a paper by Fabbri *et al.* (2015). While the first two relate to different two-sector one capital good discrete models, the last two are in continuous time. Still, the two-dimensional Leizarowitz (1985) example is framed in reduced form, while that in Fabbri *et al.* (2015), explicitly relating to a growth model, has an infinite-dimensional state space. So, while it has been already established that in discrete time non-existence is possible even with a one-dimensional state space, it is not clear which is the minimum dimension for non-existence for continuous time models[1]. We here report a new example showing that this minimum dimension is 2. In other words, our example confirms the conjecture advanced in Brock & Haurie (1976) p. 345 for optimal growth models:

> *We have not yet constructed an example where the steady state $\bar{x}$ is unique but no overtaking optimal program exists from some x^0 while a weakly overtaking optimal program exists from our x^0. Such an example will take some work to construct because it seems that the state space will have to be two dimensional whereas in discrete time as shown in Brock (1970) we can get by with a one-dimensional output space.*

2. THE MODEL

We consider the $(n+1)$-sector single-technique case of the discrete capital model introduced in Bruno (1967). In the system, there are $n+1$ commodities: n pure capital goods and a pure consumption good. The services of a primary factor of production, labor, are combined with the services of the stocks of capital to produce the $n+1$ commodities. Technology is of the discrete type, and only $n+1$ processes, one for each good, are available.

The superscript T denotes transposed matrices, $\langle \cdot, \cdot \rangle$ represents the internal product in $\mathbb{R}^n$. A unit of the j-th capital good needs to be produced a_{ij} units of the i-th capital good and ℓ_j units of labour, whereas one unit of the consumption good needs α_i units of the i-th capital good and ℓ_c units of labour, so that the technology is described by a vector and a matrix of capital coefficients $A = [a_{ij}]_{i,j=1}^n$, $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]^T$, and a vector ℓ and a scalar ℓ_c of labor input coefficients $\ell = [\ell_1, \ell_2, ..., \ell_n]^T$, ℓ_c. Let $k(t) = [k_1(t), k_2(t), ..., k_n(t)]^T$ represent the stock of capital goods at a given time $t \geq 0$, and $x(t) = [x_1(t), x_2(t); ..., x_n(t)]^T$, and $x_c(t)$ be the intensities of activation of the production processes at that time, chosen by the social planner. Assuming that the flow of new capitals is accumulated and that capitals decay at a constant depreciation rate $\delta > 0$ (the same for all capital goods), and that the initial state of the system is $k_0 \geq 0$, then the state equation is given by the n-dimensional system

$$\dot{k}(t) = -\delta k(t) + x(t),\ t \geq 0; \quad k(0) = k_0. \tag{1}$$

Assume that the labour flow available at every t is constant and normalized to 1, and that every unit of capital good instantaneously provides one unit of production services. Then the production is subject to the following set of constraints, holding for all $t \geq 0$:

$$Ax(t) + x_c(t)\alpha \leq k(t), \tag{2}$$

[1] It is known (see e.g. Example 4.1 of Leizarowitz, 1985 or Example 4.4 of Carlson *et al.*, 1991) that optimal trajectories exist for continuous time scalar systems.

(3) $\langle \ell, x(t)\rangle + x_c(t)\ell_c \le 1,$

(4) $x(t) \ge 0, x_c(t) \ge 0.$

Assuming a linear utility and a discount factor $\rho \ge 0$, the problem is that of maximizing

$$J(x, x_c, k_0) = \int_0^{+\infty} e^{-\rho t} x_c(t)\, \mathrm{d}t \tag{5}$$

over the set of admissible controls

$$\mathcal{X}(k_0) = \{(x, x_c) \in L^1_{loc}(0, +\infty; \mathbb{R}^{n+1}_+) \ : \ (1) - (4) \text{ hold at all } t \ge 0\}.$$

Remark 2.1 Since from (1) one derives $k(t) = e^{-\delta t}k_0 + \int_0^t e^{-\delta(t-s)}x(s)ds$, the solution k is in the space $W^{1,1}_{loc}(0, +\infty; \mathbb{R}^n)$, and trajectories k are always nonnegative. Moreover, if vector ℓ is strictly positive, we may define $c := \left(\sum_{i=1}^n \ell_i^{-2}\right)^{1/2}$ and check that $\|k(t)\| \le \|k_0\| + c/\delta,\ \forall t \ge 0$, that is, trajectories are uniformly bounded by a constant depending only on k_0. □

Due to (3) and (4), when $\rho > 0$ the utility is finite for all admissible controls but, on the contrary, when $\rho = 0$ it may be infinite valued. We take into consideration the following criteria of optimality.

Definition 2.2 *A control (x^*, x_c^*) in $\mathcal{X}(k_0)$ is optimal (or overtaking) at k_0 if*

$$\liminf_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\, \mathrm{d}t \ge 0$$

for every other control (x, x_c) in $\mathcal{X}(k_0)$. If k^* is the trajectory starting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is an optimal couple.

Definition 2.3 *A control (x^*, x_c^*) in $\mathcal{X}(k_0)$ is maximal (or weakly overtaking) at k_0 if*

$$\limsup_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\, \mathrm{d}t \ge 0.$$

for every other control (x, x_c) in $\mathcal{X}(k_0)$. If k^* is the trajectory starting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is a maximal couple.

Every optimal control is maximal but the viceversa is false in general.

We here list the assumptions that will be used throughout the paper.

Hypothesis 2.4
1. *The matrix A is semipositive, that is, $a_{ij} \ge 0$ for all i and j and there is at least a strictly positive element;*
2. *The vector α is semipositive, that is, $\alpha \ge 0$ and $\alpha_i > 0$ for at least one i.*
3. *The vector ℓ is positve, that is, $\ell_i > 0$ for all i; also $\ell_c > 0$.*
4. *A is indecomposable;*[2]

2.1. *Golden Rules*

The aim of this section is to define *golden rules*, that is, stationary solutions supported by stationary prices. Some properties of Hamiltonian functions will prove useful for the arguments developed afterwards. We define the *current value Hamiltonian* associated to the problem as the function

[2]In economic terms, this assumption means that each capital good enters directly or indirectly into the production of all capital goods. Since the vector α is semipositive, this also implies that each capital good enters directly or indirectly into the production of all goods. Indecomposable semipositive square matrices have some useful properties: they have a strictly positive maximal eigenvalue μ (the *Perron Frobenious eigenvalue*), right and left eigenvectors associated to this root are unique (up to multiplication by a scalar), and $(I - \lambda A)^{-1} > 0$, whenever λ is a nonnegative scalar such that $\mu\lambda < 1$ (see e. g., Kurz & Salvadori, 1995, Theorem A.3.5).

$h : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}$ given by $h(k, \lambda, x, x_c) = x_c + \langle \lambda, x - \delta k\rangle$ and the *maximal value Hamiltonian* as

$$H(k, \lambda) = \sup\{h(k, \lambda, x, x_c) \;:\; (x, x_c) \geq 0, Ax + x_c\alpha \leq k, \langle \ell, x\rangle + x_c\ell_c \leq 1\}. \tag{6}$$

The maximization process through which H is computed, corresponds to solving the following linear programming problem

$$\max[\langle \lambda, x\rangle + x_c] \tag{7}$$

subject to

$$Ax + x_c\alpha \leq k, \quad \langle \ell, x\rangle + x_c\ell_c \leq 1, \quad (x, x_c) \geq 0. \tag{8}$$

which has feasible region

$$U(k) = \{(x, x_c) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \;:\; (8) \text{ holds}\}.$$

The corresponding dual problem is

$$\min[\langle q, k\rangle + w] \tag{9}$$

subject to

$$\lambda \leq A^T q + w\,\ell, \quad 1 \leq \langle \alpha, q\rangle + w\,\ell_c, \quad q \geq 0, w \geq 0, \tag{10}$$

where $(q,w) \in \mathbb{R}^n \times \mathbb{R}$ are dual control variables having the meaning, respectively, of rental rates of capital goods and wage rate (i.e., the multiplier associated to the constraint of availability of labour). We denote the feasible region of the dual problem by

$$V(\lambda) = \{(q, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \;:\; (10) \text{ holds}\}.$$

Remark 2.5 The set $U(k)$ is nonempty and compact as a consequence of Hypothesis (2.4.4), for every given and positive k, so that the maximum is attained at some (x^*, x^*_c) and, equivalently (see e. g., Franklin, 2002, Section 1.8), there exists an optimal solution (q^*, w^*) of the corresponding dual problem, moreover

$$\begin{cases} \langle \lambda, x\rangle + x_c \leq \langle k, q\rangle + w, & \forall x, x_c, q, w \\ \langle \lambda, x^*\rangle + x^*_c = \langle k, q^*\rangle + w^*. & \end{cases} \tag{11}$$

The natural conditions of optimality associated to the problem are the following:

$$\begin{cases} \dot{k}(t) = -\delta k(t) + x(t), & t \geq 0 \\ k(0) = k_0 & \\ \dot{\lambda}(t) = (\rho + \delta)\lambda(t) - q(t), & t \geq 0 \\ (x(t), x_c(t)) \in \operatorname{argmax}\{\langle \lambda(t), x\rangle + x_c : (x, x_c) \in U(k(t))\}, & t \geq 0 \\ (q(t), w(t)) \in \arg\min\{\langle k(t), q\rangle + w : (q, w) \in V(\lambda(t))\}, & t \geq 0 \end{cases} \tag{12}$$

As a consequence of the previous remarks, we define golden rules as follows.

Definition 2.6 *A golden rule*[3] *is a stationary solution* $(\bar{k}, \bar{x}, \bar{x}_c, \bar{\lambda}, \bar{w}, \bar{q})$ *of* (12).

[3]Golden rules are sometimes called *modified* golden rules when $\rho > 0$.

3. SUFFICIENT CONDITIONS OF OPTIMALITY

We briefly discuss sufficient conditions of optimality for the problem, showing in particular that when $\rho > 0$ the golden rule is overtaking optimal, while when $\rho = 0$ it is maximal. In the last section, where the main example is presented, we will show that the golden rule is not optimal when $\rho = 0$.

3.1. *The discounted case*

Assume now that $\rho > 0$. The following theorem holds.

Theorem 3.1 *Let Hypothesis 2.4 be satisfied. Assume also $k_0 \in \mathbb{R}^n_+$, $(x^*, x^*_c) \in \mathcal{X}(k_0)$, and that there exists $\lambda^*, q^* : \mathbb{R}_+ \to \mathbb{R}^n_+$ and $w^* : \mathbb{R}_+ \to \mathbb{R}_+$, with λ^* absolutely continuous, q^* and w^* measurable and locally bounded, so that $(k^*, \lambda^*, x^*, x^*_c, q^*, w^*)$ satisfies* (12) *for almost every $t \geq 0$. If in addition*

$$\lim_{t \to +\infty} e^{-\rho t} \, \langle k^*(t), \lambda^*(t) \rangle = 0 \tag{13}$$

then $(k^*; (x^*, x^*_c))$ is an optimal couple.

The proof is standard and we omit it for brevity.

Theorem 3.2 *Assume Hypothesis 2.4. Denote by μ the Perron-Frobenius eigenvalue of A. Suppose that $\delta < \mu^{-1}$ and $0 \leq \rho < \mu^{-1} - \delta$. Then there exists a unique golden rule $(\bar{k}, \bar{x}, \bar{x}_c, \bar{\lambda}, \bar{w}, \bar{q})$, given by*

$$\begin{aligned}
&\bar{x}_c = \left[\delta\langle \ell, (I - \delta A)^{-1}\alpha\rangle + \ell_c\right]^{-1}, && \bar{x} = \delta \bar{x}_c (I - \delta A)^{-1}\alpha \\
&\bar{k} = \bar{x}_c (I - \delta A)^{-1}\alpha, && \bar{w} = \left[(\delta+\rho)\langle \alpha, \left[I - (\delta+\rho)A^T\right]^{-1} \ell\rangle + \ell_c\right]^{-1} \\
&\bar{\lambda} = \bar{w} \left[I - (\delta+\rho)A^T\right]^{-1} \ell, && \bar{q} = (\rho+\delta)\bar{\lambda}.
\end{aligned}$$

Moreover, for $\rho > 0$, $(\bar{k}, \bar{x}, \bar{x}_c)$ is optimal.

Remark 3.3 Note that the assumption $\delta < \mu^{-1}$ says that the system is vital, meaning that the production can be strictly greater than mere reproduction of capital goods after decay. As a consequence, the matrix $(I - \delta A)$ is invertible, with positive inverse $(I - \delta A)^{-1}$, as A is indecomposable. Similarly $0 \leq \rho < \mu^{-1} - \delta$ implies $(I - (\delta+\rho)A^T)$ is invertible with positive inverse $(I - (\delta+\rho)A^T)^{-1}$.

PROOF OF THEOREM 3.2: We show first that (12) is uniquely satisfied (among stationary solutions) by $(\bar{k}, \bar{x}, \bar{x}_c, \bar{\lambda}, \bar{w}, \bar{q})$. Note that the first and third equation in (12) imply $\bar{x}(t) \equiv \bar{x} = \delta \bar{k}$, and $\bar{q}(t) \equiv \bar{q} = (\rho+\delta)\bar{\lambda}$. Note also that the argmax/argmin conditions in (12) coincide with (8) (10). We then multiply the first inequality in (8) by $\bar{q}$, the second by $\bar{w}$ and sum them up

$$\langle A\bar{x}, \bar{q}\rangle + \bar{x}_c\langle \alpha, \bar{q}\rangle + \langle \ell, \bar{x}\rangle \bar{w} + \bar{x}_c \ell_c \bar{w} \leq \langle \bar{k}, \bar{q}\rangle + \bar{w}. \tag{14}$$

Similarly, we multiply the first inequality in (10) by $\bar{x}$, the second by $\bar{x}_c$ and sum them up

$$\langle \bar{\lambda}, \bar{x}\rangle + \bar{x}_c \leq \langle \bar{x}, A^T \bar{q}\rangle + \bar{x}_c\langle \alpha, \bar{q}\rangle + \langle \ell, \bar{x}\rangle \bar{w} + \bar{x}_c \ell_c \bar{w}. \tag{15}$$

By (11), the right hand side in (14) coincides with the left hand side in (15), so that all inequalities hold as equalities. As a consequence, any golden rule needs to be a solution $(\bar{k}, \bar{x}_c, \bar{\lambda}, \bar{w})$ of the following simplified system

$$\begin{cases}
\delta A\bar{k} + \bar{x}_c \alpha \leq \bar{k}, \quad \langle \delta A\bar{k} + \bar{x}_c\alpha - \bar{k}, \bar{\lambda}\rangle = 0 \\
\delta\langle \ell, \bar{k}\rangle + \bar{x}_c \ell_c \leq 1, \quad (\delta\langle \ell, \bar{k}\rangle + \bar{x}_c \ell_c - 1)\bar{w} = 0 \\
\bar{\lambda} \leq (\delta+\rho)A^T\bar{\lambda} + \bar{w}\,\ell, \quad \langle \bar{\lambda} - (\delta+\rho)A^T\bar{\lambda} - \bar{w}\,\ell, \bar{k}\rangle = 0 \\
1 \leq (\delta+\rho)\langle \alpha, \bar{\lambda}\rangle + \bar{w}\,\ell_c, \quad (1 - (\delta+\rho)\langle \alpha, \bar{\lambda}\rangle + \bar{w}\,\ell_c)\bar{x}_c = 0 \\
\bar{k} \geq 0, \quad \bar{x}_c \geq 0, \quad \bar{\lambda} \geq 0, \quad \bar{w} \geq 0.
\end{cases} \tag{16}$$

We claim that $\bar{w} > 0$. Indeed, assume by contradiction $\bar{w} = 0$ and let e_μ be the eigenvector associated with the Perron-Frobenius eigenvalue μ. Then, from the third line in (16) we derive $\langle e_\mu, \bar{\lambda}\rangle \leq \mu(\rho+\delta)\langle e_\mu, \bar{\lambda}\rangle$, which implies $1 \leq \mu(\rho+\delta)$, in contradiction with the assumptions. Now $\bar{w} > 0$ imply that the inequality in the second line of (16) holds as equality. Moreover, since (11) implies $\bar{x}_c = \bar{w} + \rho\langle\bar{\lambda}, \bar{k}\rangle$, also $\bar{x}_c > 0$, so that the fourth line of (16) holds as equality. Next we show that $\bar{k} > 0$. In fact, as $(I-\delta A)^{-1}$ is positive, the first inequality in (16) is equivalent to $\bar{k} \geq \bar{x}_c(I-\delta A)^{-1}\alpha > 0$. The fact that $\bar{k} > 0$ implies that the inequality in the third line of (16) is satisfied as equality. Then, from Remark 3.3, $\bar{\lambda} = \bar{w}(I-(\delta+\rho)A^T)^{-1}\ell$, so that also $\bar{\lambda} > 0$. As a consequence, the first line of (16) is satisfied as equality, that is $\bar{k} = \bar{x}_c(I-\delta A)^{-1}\alpha$. Summing up, the unique solution of (16) is obtained by solving as equalities the inequalities of the system, that is

$$\begin{array}{ll} \bar{k} = \bar{x}_c(I-\delta A)^{-1}\alpha, & \delta\langle \ell, \bar{k}\rangle + \bar{x}_c\ell_c = 1 \\ \bar{\lambda} = \bar{w}(I-(\delta+\rho)A^T)^{-1}\ell, & 1 = (\delta+\rho)\langle\alpha, \bar{\lambda}\rangle + \bar{w}\,\ell_c. \end{array} \tag{17}$$

which has as unique solution that described in the claim of the theorem. When $\rho > 0$, (13) is trivially satisfied by $(\bar{k}, \bar{\lambda})$, and the golden rule is optimal for Theorem 3.1. *Q.E.D.*

3.2. *The undiscounted case*

Throughout this subsection we assume $\rho = 0$. In this case, the application of the results by Brock & Haurie (1976) (see also Carlson *et al.*, 1991, chapter 4) provide the existence of a maximal couple starting at a k_0 from which the steady state $\bar{k}$ can be reached in finite time. Some preliminary work is needed.

Proposition 3.4 *Assume $\rho = 0$. The Hamiltonian H defined by (6) has a unique saddle point at $(\bar{k}, \bar{\lambda})$, in particular $H(\cdot, \bar{\lambda})$ has a maximum at $\bar{k}$, and $H(\bar{k}, \cdot)$ has a miminum at $\bar{\lambda}$.*

The proof is standard and we omit it for brevity. Now, we define the compact, convex, and possibly empty set $V(k, v) = \{(x, x_c) : (x, x_c) \in U(k), x = \delta k + v\}$, and

$$\mathcal{L}(k, v) = \begin{cases} \max\{x_c : (x, x_c) \in V(k, v)\} & V(k, v) \neq \emptyset \\ -\infty & V(k, v) = \emptyset \end{cases}$$

and the value-loss function as

$$\theta(k, v) = \mathcal{L}(\bar{k}, 0) - \mathcal{L}(k, v) - \langle\bar{\lambda}, v\rangle\,. \tag{18}$$

Remark 3.5 Lemma 4.3 in Carlson *et al.* (1991) implies $\mathcal{L}(k, v)$ is concave in both variables. Moreover,[4] the arguments at pages 64-65 (in particular (4.84)) there contained imply $\theta(k, v) \geq 0$ for all (k, v) in $\mathbb{R}^n \times \mathbb{R}^n$.

The original problem is paired with the associated Lagrange Problem (briefly, ALP) of minimizing the integral of the value-loss function along $(k(t), \dot{k}(t))$. A solution is defined as an absolutely continuous function $k^* : [0, \infty) \to \mathbb{R}^n$, such that $k(0) = k_0$, and

$$\liminf_{T\to\infty} \int_0^T \Big[\theta(k(t), \dot{k}(t)) - \theta(k^*(t), \dot{k}^*(t))\Big]\, dt \geq 0. \tag{19}$$

Theorem 3.6 *Assume $k_0 \in \mathbb{R}^n_+$, and that $\bar{k}$ is reachable from k_0, along an admissible trajectory, in finite time. Then:*

(*i*) *there exists a solution of the ALP;*

(*ii*) *all solutions of ALP are maximal trajectories for the original problem. In particular the golden rule is a maximal solution.*

[4] Note that the golden rule stock is also the unique stationary solution of the maximization problem (4.34) defined in Carlson *et al.* (1991), moreover our H coincides with the Hamiltonian $\mathcal{H}$ there defined in (4.81), $\mathcal{H}(k, \lambda) = \sup_{v\in\mathbb{R}^n}\{\mathcal{L}(k, v) + \langle\lambda, v\rangle\}$.

PROOF: The proof of (i) follows from Theorem 4.7 in Carlson *et al.* (1991), as $\mathcal{L}(k,v)$ is concave, and $(\bar{k},\bar{\lambda})$ is a saddle point for the Hamiltoninan H. Moreover the set of velocities $\varphi(k) = \{x - \delta k : (x, x_c) \in U(k)\}$ is a compact and convex set. The proof of (ii) can be deduced from the proof of Theorem 4.9 p.69, where the fact is shown under Assumption 4.5 (and not 4.4 as erroneously reported there) p.64. *Q.E.D.*

Good controls defined below are important as they yield a finite integral of value-loss.

Definition 3.7 *A control (x, x_c) is good if the associated trajectory k satisfies*

$$\liminf_{T\to\infty} \int_0^T \left[\mathcal{L}(k(t),\dot{k}(t)) - \mathcal{L}(\bar{k},0)\right] dt > -\infty. \tag{20}$$

Remark 3.8 Observe that $\int_0^T \mathcal{L}(k(t),\dot{k}(t)) - \mathcal{L}(\bar{k},0)dt$ equals $\int_0^T -\theta(k(t),\dot{k}(t))dt + \langle\bar{\lambda}, k(0) - k(T)\rangle$. Since Remark 2.1 implies $k(T)$ is uniformly bounded in T, and Remark 3.5 implies $\theta(k(t),\dot{k}(t)) \geq 0$ for any t, the condition (20) is verified if and only if the following milder condition holds: $\limsup_{T\to\infty} \int_0^T \mathcal{L}(k(t),\dot{k}(t)) - \mathcal{L}(\bar{k},0)dt > -\infty$.

Lemma 3.9 *Assume that $k_0 \in \mathbb{R}^n_+$ is such that there exists an admissible control stirring k_0 to $\bar{k}$ in finite time. Then any maximal (in particular, optimal) control at k_0 is good.*

PROOF: Assume (x, x_c) is a maximal admissible control at k_0, and let k be the associated trajectory. Consider (y, y_c) admissible at k_0, with (y, y_c) stirring k_0 to $\bar{k}$ in a time T_0 and then coinciding with $(\bar{x}, \bar{x}_c)$ in $(T_0, +\infty)$. Then for all $T \geq T_0$:

$$\begin{aligned}\int_0^T (x_c(t) - y_c(t))\,dt &= \int_0^{T_0} (x_c(t) - y_c(t))\,dt - \int_0^{T_0} (x_c(t) - \bar{x}_c)dt + \int_0^T (x_c(t) - \bar{x}_c)\,dt\\ &\leq \int_0^{T_0} (x_c(t) - y_c(t))\,dt - \int_0^{T_0} (x_c(t) - \bar{x}_c)dt + \int_0^T \left[\mathcal{L}(k(t),\dot{k}(t)) - \mathcal{L}(\bar{k},0)\right] dt.\end{aligned}$$

The limsup as T tends to $+\infty$ of the left hand side is nonnegative, as (x, x_c) is maximal, and the first two addenda in the right hand side are bounded. As a consequence, $(k(t),\dot{k}(t))$ satisfies (20) in view of Remark 3.8.

Q.E.D.

4. THE EXAMPLE OF NONEXISTENCE

We introduce the following example and study the behaviour of specific solutions both in the discounted and undiscounted case. We set

$$n = 2,\ \delta = \frac{2}{3},\ \ell_c = 1,\ A = \begin{bmatrix} 1/2 & 3/4 \\ 1/4 & 7/8 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} \quad \ell = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{21}$$

so that Hypothesis 2.4 is verified, and A has eigenvalues $(\pm\sqrt{57} + 11)/16$, with $\mu^{-1} > 2/3$. Theorem 3.2, implies the golden rule is (independent of ρ and) given by

$$\bar{x}_c = \frac{2}{9}, \quad \bar{x} = \begin{bmatrix} \frac{23}{63} \\ \\ \frac{26}{63} \end{bmatrix}, \quad \bar{k} = \begin{bmatrix} \frac{23}{42} \\ \\ \frac{13}{21} \end{bmatrix}. \tag{22}$$

with associated support prices depending on ρ:

$$\bar{w} = \frac{72\rho^2 - 300\rho + 56}{81\rho^2 + 252}, \quad \bar{\lambda} = \begin{bmatrix} \frac{56-60\rho}{27\rho^2+84} \\ \\ \frac{24\rho+112}{27\rho^2+84} \end{bmatrix}, \quad \bar{q} = \frac{2+\rho}{3} \begin{bmatrix} \frac{56-60\rho}{27\rho^2+84} \\ \\ \frac{24\rho+112}{27\rho^2+84} \end{bmatrix} \tag{23}$$

Now consider system (1) and choose the admissible controls that satisfies (2) (3) as equalities. By inverting those relations, one obtains

$$\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} A & \alpha \\ \ell^T & \ell_c \end{bmatrix}^{-1} \begin{bmatrix} k(t) \\ 1 \end{bmatrix} \tag{24}$$

which substituted into (1) imply

$$\begin{cases} \dot{k}_1(t) = -\frac{2}{9}k_1(t) - \frac{16}{9}k_2(t) + \frac{11}{9} \\ \dot{k}_2(t) = \frac{16}{9}k_1(t) + \frac{2}{9}k_2(t) - \frac{10}{9}. \end{cases} \tag{25}$$

The system has purely imaginary eigenvalues so that one obtains the periodic solutions

$$\begin{cases} \hat{k}_1(t) = c_1 \cos\frac{2\sqrt{7}t}{3} - c_2 \sin\frac{2\sqrt{7}t}{3} + \frac{23}{42} \\ \hat{k}_2(t) = \frac{1}{8}\left(c_2 + 3\sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{1}{8}\left(3\sqrt{7}c_2 - c_1\right)\cos\frac{2\sqrt{7}t}{3} + \frac{13}{21} \end{cases} \tag{26}$$

where the constants c_1 and c_2 depend on $k_0 = (k_1^0, k_2^0)$:

$$k_1^0 - \bar{k}_1 = c_1, \qquad k_2^0 - \bar{k}_2 = \frac{3\sqrt{7}}{8}c_2 - \frac{1}{8}c_1. \tag{27}$$

The associated controls $(\hat{x}, \hat{x}_c)$ can also be computed by means of (24):

$$\begin{cases} \hat{x}_1(t) = \frac{1}{3}(c_1 - 2\sqrt{7}c_2)\cos\frac{2\sqrt{7}t}{3} - \frac{1}{3}\left(2c_2 + 2\sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{23}{63} \\ \hat{x}_2(t) = \frac{1}{3}\left(5c_1 + \sqrt{7}c_2\right)\cos\frac{2\sqrt{7}t}{3} + \frac{1}{3}\left(\sqrt{7}c_1 - 5c_2\right)\sin\frac{2\sqrt{7}t}{3} + \frac{26}{63} \\ \hat{x}_c(t) = \frac{1}{3}(\sqrt{7}c_2 - 7c_1)\cos\frac{2\sqrt{7}t}{3} + \frac{1}{3}\left(7c_2 + \sqrt{7}c_1\right)\sin\frac{2\sqrt{7}t}{3} + \frac{2}{9}. \end{cases} \tag{28}$$

Note that (27) imply that c_1 and c_2 are small for small differences of k_0 from $\bar{k}$. As a consequence, for c_1 and c_2 small enough: a) the whole trajectory is contained in a ball, centered at $\bar{k}$ and of arbitrarily small radius; b) $(\hat{x}, \hat{x}_c)$ is also cycling at an arbitrarily small neighborhood of $(\bar{x}, \bar{x}_c)$; c) since for $k_0 = \bar{k}$ the trajectory $k(t) \equiv \bar{k}$ satisfies strictly the positivity constraints (4), that remains true also for k_0 close enough to $\bar{k}$; hence, the constraints on the associated dual variables (the argmin condition in (12)) hold unchanged, and support prices associated to $\hat{k}, \hat{x}, \hat{x}_c$ coincide with $\bar{\lambda}, \bar{q}, \bar{w}$.

Assume now $\rho > 0$. As a consequence of the previous arguments, for k_0 close enough to $\bar{k}$, $(\hat{k}, (\hat{x}, \hat{x}_c), \bar{\lambda}, \bar{q}, \bar{w})$ satisfy the assumptions of Theorem 3.1 and is hence optimal. We have then proved the following result.

Proposition 4.1 *Let $k_0 > 0$. The system (1)(2)(3)(4), with data (21), has a periodic solution $(\hat{k}, (\hat{x}, \hat{x}_c))$ given by (26)(28). For $\rho > 0$ and for k_0 sufficiently close to $\bar{k}$, the admissible couple $(\hat{k}, (\hat{x}, \hat{x}_c))$ is optimal at k_0, and it is supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$.*

Now we prove that, in the specific case here described, any initial condition $k_0 > 0$ can be driven to the steady state $\bar{k}$ in finite time by means of an admissible control.

Lemma 4.2 *Let $k_0 \in \mathbb{R}^2$, $k_0 > 0$. Then there exists $T(k_0) \geq 0$ and a control $(\tilde{x}, \tilde{x}_c) \in \mathcal{X}(k_0)$ such that the associated trajectory $\tilde{k}(t)$ starting at k_0 reaches $\bar{k}$ at time $T(k_0)$.*

PROOF: We first consider the case in which $k_0 = \gamma_0 \bar{k}$, for a $\gamma_0 > 0$. If $\gamma_0 = 1$, there is nothing to prove. If $\gamma_0 > 1$, we choose $\tilde{x}(t) = 0$, $\tilde{x}_c(t) = 0$, for all $t \geq 0$, so that the constraints are trivially satisfied. With the choice $T(k_0) = \delta^{-1} \ln \gamma_0$, we obtain $\tilde{k}(T(k_0)) = \bar{k}$.

If instead $\gamma_0 < 1$, we choose $\tilde{x}_c(t) = 0$, and $\tilde{x}(t) = g\tilde{k}(t)$, for all $t \geq 0$.] As it can be shown by direct proof, with the choices $T(k_0) = (g - \delta)^{-1} \ln(1/\gamma_0)$, $\quad (2/3) < g \leq (23/31)$ the pair $(\tilde{k}, (\tilde{x}, \tilde{x}_c))$ is admissible and $\tilde{k}(T(k_0)) = \bar{k}$.

Now assume $k_0 \notin \{\gamma\bar{k} : \gamma_0 \in \mathbb{R}^+\}$, $k_0 = (k_{01}, k_{02})$ and that, for instance, $k_{02}/k_{01} > \bar{k}_2/\bar{k}_1$. We define (y, y_c), and the associated trajectory k^y, as follows $y_c(t) = 0$, $y_1(t) = gk_1^y(t)$, and $y_2(t) =$

0, for all $t \geq 0$, with g a positive constant. The trajectory k^y reaches $\{\gamma\bar{k} : \gamma \in \mathbb{R}^+\}$ at time $T_0 = (1/g)\ln\left(k_{02}\,\bar{k}_1(k_{01}\,\bar{k}_2)^{-1}\right) > 0$, and the chosen control (y, y_c) is admissible when $0 < g \leq \min\left\{\delta,\ (a_{11})^{-1}, \bar{k}_2(\bar{k}_1 a_{21})^{-1}, (\ell_1 k_{10})^{-1}\right\}$. Once on $\{\gamma\bar{k} : \gamma \in \mathbb{R}^+\}$, we may stir the trajectory to $\bar{k}$ by making use of the control $(\hat{x}, \hat{x}_c)$ built in the first part of the proof, pulled back of the time T_0, and reach the steady state in time $T(k_0) = T_0 + T(k^y(T_0))$. *Q.E.D.*

4.1. *The undiscounted case*

We use the previous example to show that the golden rule given by Definition 2.6 may fail to be optimal when the discount ρ is null. More in general, we will prove that when $\rho = 0$ the cycles described by (26)(28) are maximal but fail to be optimal for all k_0 close enough to $\bar{k}$, and derive as a particular case that the golden rule $(\bar{k}, \bar{x}, \bar{x}_c)$ is maximal and not optimal at $\bar{k}$.

Remark 4.3 Regardless the initial condition, when $\rho = 0$ the utility yielded by the control $\hat{x}_c(t)$ described in (28) in a time interval of a period length equals the utility yielded by $\hat{x}_c$ in the same time span. Note that the period of the cycle is $P = 3\pi/\sqrt{7}$, moreover

$$\int_\sigma^{\sigma+P} \hat{x}_c(t)dt = \int_\sigma^{\sigma+\frac{3\pi}{\sqrt{7}}} \bar{x}_c dt = \frac{2\sqrt{7}}{21}\pi, \quad \forall\sigma \geq 0.$$

Lemma 4.4 *For all initial capital stocks $k_0 \in \mathbb{R}^n_+$, there exists a maximal couple starting at k_0. In particular, the cycles described by (26)(28) and the golden rule $(\bar{k}, \bar{x}, \bar{x}_c)$ are maximal.*

Proof: In order to apply Theorem 3.6 it is enough to show that cycles described by (26) are minimizers of the integral of losses described in (19). Note that

$$\theta(\hat{k}(t), \dot{\hat{k}}(t)) = \bar{x}_c - \hat{x}_c(t) - \left\langle \bar{\lambda}, \hat{x}(t) - \delta\hat{k}(t) \right\rangle.$$

Moreover, since $(\hat{k}, \hat{x}, \hat{x}_c)$ are supported by the same prices $\bar{\lambda}, \bar{q}, \bar{w}$ of the golden rule, (11) implies

$$\bar{x}_c + \langle\bar{\lambda}, \bar{x}\rangle - \left[\langle\bar{k}, \bar{q}\rangle + \bar{w}\right] = 0 = \hat{x}_c(t) + \langle\bar{\lambda}, \hat{x}(t)\rangle - \left[\left\langle\hat{k}(t), \bar{q}\right\rangle + \bar{w}\right]$$

Subtracting the third term from the first term in the previous equalities, and recalling that $\bar{q} = \delta\bar{\lambda}$ and $\bar{x} - \delta\bar{k} = 0$, one has

$$0 = \bar{x}_c - \hat{x}_c(t) - \left\langle \bar{\lambda}, \hat{x}(t) - \delta\hat{k}(t) \right\rangle = \theta(\hat{k}(t), \dot{\hat{k}}(t)). \tag{29}$$

Then cycles $\hat{k}$ are minimizers for ALP and $(\hat{k}, \hat{x}, \hat{x}_c)$ is a maximal couple. *Q.E.D.*

Theorem 4.5 *There exists a neighborhood U of $\bar{k}$ in $\mathbb{R}^n_+$, such that for all k_0 in U the cycles $(\hat{k}, \hat{x}, \hat{x}_c)$ starting at k_0 and described by (26)(28) are not optimal at k_0. Moreover, with data (21), there is no admissible control which is optimal at such k_0.*

From the previous Theorem, we derive the following corollary.

Corollary 4.6 *The golden rule $(\bar{k}, \bar{x}, \bar{x}_c)$ is not optimal at $\bar{k}$, and there is no admissible control which is optimal at $\bar{k}$.*

Proof of Theorem 4.5: We start by proving that the cycles $(\hat{k}, (\hat{x}, \hat{x}_c))$ starting at k_0 are not optimal, by building a control (y, y_c) not overtaken by $(\hat{x}, \hat{x}_c)$. Assume $\theta > 0$ is such that for all $k_0 \in B(\bar{k}, 2\theta)$ the cycles described by (26)(28) starting at k_0 are supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$, then select $k_0 \in B(\bar{k}, \theta)$. For an arbitrarily chosen $\tau > 0$, set

$$\begin{cases} y(t) = 0, \quad y_c(t) = \bar{x}_c + 8\theta, & t \in [0, \tau] \\ y(t) = \hat{x}(t-\tau), \quad y_c(t) = \hat{x}_c(t-\tau), & t \in (\tau, +\infty). \end{cases}$$

By explicit calculations, one may see that there exist positive θ_1 and τ_1 such that for all $0 < \theta < \theta_1$ and $0 < \tau < \tau_1$ the constraints (2) (3) (4) are satisfied in $[0, \tau]$ (for instance, $\tau_1 = 3/2$, $\varepsilon_1 = (26 - 7e)[42\,(1 + 6e)]^{-1}$). We assume also $\tau < \tau_2$ where $\tau_2 > 0$ is such that $\|k^y(\tau_2) - k_0\| < \theta$, so that the cycle starting at $k^y(\tau)$ and described by (26)(28) is supported by stationary prices $(\bar{\lambda}, \bar{q}, \bar{w})$. Note that (27) and (28) imply $\int_0^\tau y_c(t)dt > \int_0^\tau \hat{x}_c(t)dt + \tau\theta$. By direct calculation one obtains (see also Remark 4.3, where P is defined) that for all $n \in \mathbb{N}$

$$\int_0^{\tau+nP} (y_c(t) - \hat{x}_c(t))dt \geq \tau\theta > 0, \tag{30}$$

so that $(\hat{x}, \hat{x}_c)$ cannot be optimal.
Now we show that no admissible strategy can be optimal at k_0. Assume by contradiction that there exists $(\tilde{x}, \tilde{x}_c) \in \mathcal{X}(k_0)$ optimal at k_0. Then for $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt \geq -\varepsilon, \text{ and } \int_0^T (\tilde{x}_c(t) - y_c(t))dt \geq -\varepsilon, \text{ for all } T \geq T_\varepsilon. \tag{31}$$

Since $T \mapsto \int_0^T (y_c(t) - \hat{x}_c(t))dt$, is continuous and (30) holds, there exists $\upsilon > 0$ such that

$$\int_0^T (y_c(t) - \hat{x}_c(t))dt \geq \frac{\tau\theta}{2}, \text{ for all } T \in [\tau, \tau + \upsilon].$$

Set $T_n = \tau + nP$, and $n_\varepsilon = \min\{n \in \mathbb{N} : T_n > T_\varepsilon\}$. Then, by periodicity

$$\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt = \int_0^T (y_c(t) - \hat{x}_c(t))dt + \int_0^T (\tilde{x}_c(t) - y_c(t))dt \geq \frac{\tau\theta}{2} - \varepsilon, \tag{32}$$

for any $T \in [T_n, T_n + \upsilon]$, $n \geq n_\varepsilon$. We show first that

$$\liminf_{n\to\infty} \frac{1}{T_n + \upsilon} \int_0^{T_n+\upsilon} \left(\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt \right) dT \geq \frac{\theta\tau\upsilon}{4P} \tag{33}$$

Note that, if $* = \int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))dt$, one may split the previous integral as follows, and use (31) and (32) to derive

$$\begin{aligned} \int_0^{T_n+\upsilon} (*)\, dT &= \int_0^{T_{n_\varepsilon}} (*)\, dT + \sum_{i=n_\varepsilon}^{n} \int_{T_i}^{T_i+\upsilon} (*)\, dT + \sum_{i=n_\varepsilon}^{n-1} \int_{T_i+\upsilon}^{T_{i+1}} (*)\, dT \\ &\geq \int_0^{T_{n_\varepsilon}} (*)\, dT + (n - n_\varepsilon + 1) \left(\frac{\tau\theta}{2} - \varepsilon \right) \upsilon - (P - \upsilon)\varepsilon(n - n_\varepsilon) \end{aligned} \tag{34}$$

so that

$$\frac{1}{T_n + \upsilon} \int_0^{T_n+\upsilon} (*)\, dT \geq \frac{\theta\tau\upsilon}{2P} - \varepsilon + o\left(\frac{1}{n}\right)$$

with $o(1/n)$ tending to 0, as n tends to $+\infty$. Then (33) holds when $\varepsilon \leq (\theta\tau\upsilon)/(4P)$. On the other hand, Remark 3.5 and (29) imply $\tilde{x}_c(t) - \hat{x}_c(t) \leq \langle \bar{\lambda}, \dot{\hat{k}}(t) - \dot{\tilde{k}}(t) \rangle$ so that

$$\frac{1}{S} \int_0^S \left(\int_0^T (\tilde{x}_c(t) - \hat{x}_c(t))\, dt \right) dT \leq \langle \bar{\lambda}, \frac{1}{S} \int_0^S \left(\hat{k}(T) - \tilde{k}(T) \right) dT \rangle \tag{35}$$

Note that both $(\hat{x}, \hat{x}_c)$ and $(\tilde{x}, \tilde{x}_c)$ are good controls in the sense of Definition 3.7 (the first by the direct proof given in Lemma 4.4, the second by Lemma 3.9). Hence, by Lemma 4.7 p.69 in Carlson *et al.* (1991) (see also Definition 3.7 and the remarks below) one has $\lim_{S\to\infty} \int_0^S \hat{k}(T)dT = \lim_{S\to\infty} \int_0^S \tilde{k}(T)dT = \bar{k}$, so that passing to limits in (35) and comparing with (33) we derive a contraddiction. *Q.E.D.*

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